with L. Göttsche, K. Yoshioka

X : C^∞ 4-mfd, $b^+>1$, $b_1=0$

$(K_X^2) := 2\chi(X)+3\sigma(X)$ $\quad \chi_h(X) := \dfrac{\chi(X)+\sigma(X)}{4}$

▷ Donaldson invariants

$$\mathcal{D}^{\mathfrak{z}}(\exp(\alpha z+px)) := \sum_y \Lambda^{\dim M(y)} \int_{M(y)} \exp(\mu(\alpha z+px))$$

z, x : variable $\quad \alpha\in H_2(X)$, $p=pt\in H_0(X)$

$y=(2,\mathfrak{z},n)\in H^*(X)$ $\quad$ $\mathfrak{z}$: fix, but move n

$M(y)$: moduli of $U(2)$-instanton with Chern class $=y$

$\mathcal{E}$: universal b'dle on $X\times M(y)$

$$\mu(\alpha z+px) = \int_X \left(c_2(\mathcal{E})-\tfrac14 c_1(\mathcal{E})^2\right)\cup(\alpha z+px)$$

(Def. X : KM-simple type $\iff (\frac{\partial^2}{\partial x^2}-4\Lambda^2)\mathcal{D}^{\mathfrak{z}}=0$

▷ Seiberg-Witten invariants

$\mathfrak{s}$: spin^c str. $\quad c_1(\mathfrak{s})=c_1(S^+)$ $\rightsquigarrow SW(\mathfrak{s})\in\mathbb{Z}$

Def. X : SW simple type $\iff$ SW inv. $\neq 0$ only if v.dim. of moduli sp. $=0$

$SW(\mathfrak{s})$: SW invariant $\overset{\text{def.}}{\iff} c_1(\mathfrak{s})^2=(K_X^2)$

no non-simple type 4-mfd example found

Witten's conjecture (1994)

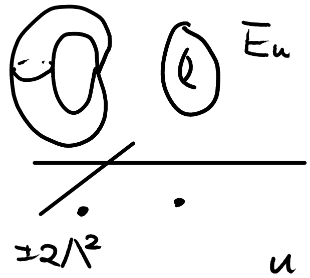
X : SW simple type

$$\Longrightarrow \mathcal{D}^{\mathfrak{z}}(\exp\alpha\times(1+\tfrac12 p)) = 2^{(K_X^2)-\chi_h(X)+2}(-1)^{\chi_h(X)}$$

$$\times e^{(\alpha^2)/2}\sum_{\mathfrak{s}} SW(\mathfrak{s})(-1)^{(\mathfrak{z},\mathfrak{z}+c_1(\mathfrak{s}))/2}e^{(c_1(\mathfrak{s}),\alpha)}$$

(finite sum)

SW curves : $y^2 = 4x(x^2 + ux + \Lambda^4)$



$u = \infty$ ---- $\mathcal{D}$

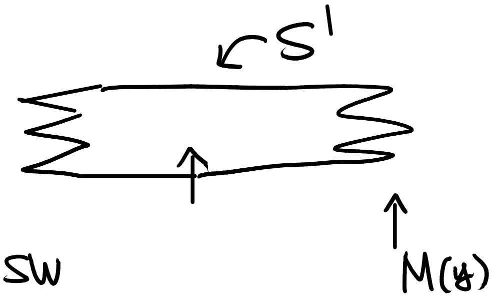
$u = \pm 2\Lambda^2$ ---- elliptic curve is singular

$\Longrightarrow$ SW contribution

mathematical approach

- Pidstrigach – Tyurin
- Feehan – Leness

SO(3)-monopole cobordism (with a fund. matter)



$\Longrightarrow$ Witten's conj. when $(K_X^2) \geq \chi_h(X) - 3$ or abundant

computation of local contribution around fixed pts difficult because of singularity

- Mochizuki : X : cpx proj. surface

explicit formula of local contribution in terms of integration over Hilb. scheme of pts on X

<u>GNY</u> : 1°. enough to compute X : toric surface

2°. localization $\Longrightarrow$ Nekrasov partition func.

$X = \mathbb{R}^4$

○ partition function for $N=2$ $SU(2)$ SUSY YM with a single fund. matter

$M(n) = M(2,n)$

$\curvearrowleft T^3$ $\quad \mathrm{Lie}\, T^3 = \mathbb{C}\varepsilon_1 \oplus \mathbb{C}\varepsilon_2 \oplus \mathbb{C}a$ (after Nekrasov)

matter bdle $\quad \mathcal{V}_{(E,\varphi)} = H^1(E(-l_\infty)) \otimes \underbrace{K_{\mathbb{C}^2}^{1/2}}_{\approx\, e^{-\varepsilon_1+\varepsilon_2/2}}$

$\curvearrowleft S^1$ multiplication

$\mathrm{Lie}\, S^1 = \mathbb{C}m$ (matter)

$$Z^{in}(\varepsilon_1,\varepsilon_2,a,m,\Lambda) := \sum_n \Lambda^{3n} \int_{M(n)} e(\mathcal{V}\otimes e^m)$$

— definition by localization

fixed pts … pair of Young diagram

$$= \sum_{\vec{Y}=(Y^1,Y^2)} \frac{e(H^1(I_{Y_1}(-l_\infty)\otimes e^m)\, e(H^1(I_{Y_2}(-l_\infty)\otimes e^m)}{\prod_{\alpha,\beta=1,2} e(\mathrm{Ext}^1(I_{Y_\alpha}, I_{Y_\beta}(-l_\infty))\otimes e^{a_\beta - a_\alpha})}$$

$(a_2 = a,\ a_1 = -a)$

Rem • pure theory : replace $e(\mathcal{V}\otimes e^m)$ by 1

Prop. (1) $\varepsilon_1\varepsilon_2 \log Z^{in}(\varepsilon_1,\varepsilon_2,a,m,\Lambda)$ is regular at $\varepsilon_1=\varepsilon_2=0$

$$=: F_0^{in}(a,m,\Lambda) + (\varepsilon_1+\varepsilon_2)\times H^{in}(a,m,\Lambda)$$

$$+ \underbrace{\varepsilon_1\varepsilon_2}_{=\chi(\mathbb{C}^2)} A^{in}(a,m,\Lambda) + \underbrace{\frac{\varepsilon_1^2+\varepsilon_2^2}{3}}_{=\sigma(\mathbb{C}^2)} B^{in}(a,m,\Lambda) + \text{higher}$$

(2) $H^{in} = 0$

Th. $\mathcal{D}^{\xi}(\exp(\alpha z + p x))$

$$= \sum_{\mathfrak{s}} SW(\mathfrak{s}) \operatorname*{Res}_{a=\infty} \mathcal{B}(\mathfrak{s}, \xi; a)\, da$$

where

$$\mathcal{B}(\mathfrak{s}, \xi; a)\, da = \frac{da}{a} (-1)^{\star} 2^{\star\star} \qquad \xi' := c_1(\mathfrak{s}) - (\xi - K_X)$$

$$\left(\frac{2a}{\Lambda}\right)^{((\xi - K_X)^2) + (K_X^2) + 3\chi_h(X) - 2(\xi - K_X, c_1(\mathfrak{s}))} \exp\left(-(\xi - K_X - c_1(\mathfrak{s}), \alpha) a z - a^2 x\right)$$

$$\exp\Big[\frac{1}{3} \frac{\partial F_0^{in}}{\partial \log \Lambda} x + \Big(\frac{1}{8} \frac{\partial^2 F_0^{in}}{\partial a^2} + \frac{1}{4} \frac{\partial^2 F_0^{M}}{\partial a \partial m} + \frac{1}{8} \frac{\partial^2 F_0^{in}}{\partial m^2}\Big)((\xi - K_X)^2)$$

$$- \frac{1}{4}\Big(\frac{\partial^2 F_0^{in}}{\partial a \partial m} + \frac{\partial^2 F_0^{in}}{\partial a^2}\Big)(\xi - K_X, c_1(\mathfrak{s}))$$

$$+ \frac{1}{6}\Big(\frac{\partial^2 F_0^{in}}{\partial a \partial \log \Lambda} + \frac{\partial^2 F_0^{in}}{\partial m \partial \log \Lambda}\Big)(\xi - K_X, \alpha) z - \frac{1}{6} \frac{\partial^2 F_0^{in}}{\partial a \partial \log \Lambda}(c_1(\mathfrak{s}), \alpha) z$$

$$+ \frac{1}{18} \frac{\partial^2 F_0^{in}}{\partial \log \Lambda^2} (\alpha^2) z^2 + \chi_h(X)(12 A^{in} - 8 B^{in})$$

$$+ (K_X^2)\Big(B^{in} - A^{in} + \frac{1}{8} \frac{\partial^2 F_0^{in}}{\partial a^2}\Big)\Big]$$

evaluated at $(a, m = a, \Lambda^{4/3} a^{-1/3})$

Conjecture

This is also true for C^∞-4 mfd X

K_X ⋯ spin^c structure

From now we assume the conjecture

Next we compute the partition function via SW curve.

Key definitions

- $\tau := -\dfrac{\partial^2 F_0^{in}}{\partial a^2} + \dfrac{1}{2\pi\sqrt{-1}}\left(-8\log\dfrac{\sqrt{-1}(a_2-a_1)}{\Lambda} + \log\dfrac{(a_1+m)(a_2+m)}{\Lambda}\right)$
- $u := a^2 - \dfrac{1}{3}\dfrac{\partial F_0^{in}}{\partial \log\Lambda}$
- $\omega := -2\pi\sqrt{-1}\left(\dfrac{\partial u}{\partial a}\right)^{-1}$, $\omega' = \tau\omega$

consider $E_\tau = \mathbb{C}/\mathbb{Z}\omega\oplus\mathbb{Z}\omega'$ elliptic curve (SW curve)

and the associated $\wp$-function, σ-function etc

Th. 1) E_τ is given by

$$y^2 = 4x^2(x+u) + 4m\Lambda^3 x + \Lambda^6$$

$$(\text{i.e. } x = \wp(z),\ y = \wp(z'))$$

$$\left(\Rightarrow\ \omega = \int_A \frac{dx}{y},\quad \omega' = \int_B \frac{dx}{y}\right)$$

2) All other necessary derivatives are given by elliptic integrals etc.

e.g. $$\exp A^{in} = \left(-\frac{2\sqrt{-1}a}{\Lambda}\right)^{-1/2} \times \left(-\frac{\sqrt{-1}}{\Lambda}\frac{\partial u}{\partial a}\right)^{1/2} \quad \text{etc}$$

Rem. This is the SW curve for the theory with <u>matter</u>, not pure theory

Agyres-Douglas point $a=m$

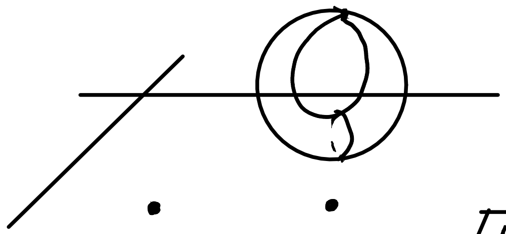
At AD point, SW curve = rational

so elliptic integrals can be given by trigonometric functions.

$\Rightarrow$ everything becomes explicit.

$$\tau := -\frac{\partial^2 F_0^{in}}{\partial a^2} + \frac{1}{2\pi\sqrt{-1}}\left(-8\log\frac{\sqrt{-1}(a_2-a_1)}{\Lambda} + \log\frac{(a_1+m)(a_2+m)}{\Lambda}\right)$$

$a=m \Rightarrow a_1+m=0 \qquad q=e^{\pi i\tau}=0$!



$u=\pm 2\Lambda^2$

$\phi=1$

$\sqrt{u}$ or ϕ

Th. $B(\$,\xi,a)\,da$ analytically continued to a meromorphic differential over

$$\phi = \frac{1}{\Lambda}\sqrt{\frac{1}{3}\left(u+\left(\frac{\pi}{\omega}\right)^2\right)} \quad \text{(contact term)}$$

$u=\infty$ ---- $\phi=0$

$u=\pm 2\Lambda^2$ ---- $\phi^4=1$

$$\operatorname*{Res}_{\phi=\sqrt[4]{1}} B(\$,\xi,a)\,da = \text{Witten's conjecture}$$

But we get a new singular pt $\phi=\sqrt[4]{1/3}$

"superconformal pt"

both A, B cycles shrinks to 0

Ref (Mariño-Moore-Peradze)

Assume X: SW simple type

X : superconformal simple type

$\Longleftrightarrow$ a) $(K_X^2) \geq \chi_h(X) - 3$

def. or b) $\sum_{\mathfrak{s}} (-1)^{(K_X, K_X + c_1(\mathfrak{s}))/2} SW(\mathfrak{s}) (c_1(\mathfrak{s}), \alpha)^n = 0$

$$0 \leq^{\forall} n \leq \chi_h(X) - (K_X^2) - 4$$

e.g. X: elliptic surface

Th. 1) X: superconformal simple type

2) $\operatorname{Res}_{\phi = \sqrt[4]{1/3}} \mathcal{B}\, da = 0 \Rightarrow$ Witten's conj.

proof of 1)

$\operatorname{Res}_{\phi = \sqrt[4]{1/3}} \mathcal{B}\, da$ apriori depends on ξ

But it must be independent (up to sign)

$\xi \mapsto \xi + 2\eta$

(twist by a line bdle)

$\Rightarrow$ nontrivial constraint